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Optimal Linear Regulators with Exponentially Time-Weighted Quadratic Performance Indices

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Introduction

ONE of the performance indices used in the classical design of single-input and single-output linear time-invariant systems is the integral of exponentially time-weighted squared error. This type of performance indices heavily penalizes long-duration errors and hence tends to give designs that yield well-damped responses. The use of exponentially time-weighted performance indices has been in later years revived by Kalman et al.,[1] Tyler,[2] and Sage[3] in the design of optimal linear regulators. The solution of this optimal linear regulator problem for finite terminal time is straightforward[2,3] and differs only slightly from that of the conventional problem. The solution for this particular problem for infinite terminal time, however, is not as trivial and is not available in literature. It is the intent of this Note to present a solution for the linear regulator problem optimal for exponentially time-weighted quadratic performance indices. The method of solution utilizes the concept of cost equivalence, to be defined in the next section, in conjunction with the theory of optimal linear regulator.

Concept of Cost Equivalence

For the sake of convenience, a linear time-invariant asymptotically stable system $\dot{y} = Fy$ and its associated quadratic cost functional

$$J = \int_0^\infty y'Pydt$$

will be denoted by $[F,P]$. The representation $[F_1,P_1]$ is said to be cost-equivalent to $[F_2,P_2]$ if F_1 and F_2 are stability matrices, P_1 and P_2 are nonnegative symmetric matrices and $J_1 = J_2$. Based on this notation and definition, the following result can be deduced immediately without proof.

Lemma: $[F,e^{2\lambda t}P]$ is cost equivalent to $[F + \lambda I,P]$ for any λ, if both representations possess the same initial conditions, and the equivalence is one-to-one correspondent.

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It should be emphasized that, although the two representations are cost-equivalent, they are not equivalent in the sense of Lyapunov,[4] and their trajectories are related by the nonsingular transformation $e^{\lambda t}I$. Since the optimal linear regulator problem is concerned primarily with the minimization of the cost of a system rather than the shape of the trajectory of the system, the concept of cost equivalence will be useful in the development of the main result of the Note.

Main Result

Consider a linear time-invariant controllable system

$$\dot{x} = Ax + Bu, \; x(0) = x_0 \tag{1}$$

and a quadratic cost functional

$$J = \int_0^\infty e^{2\sigma t}(x'Qx + u'Ru)dt \tag{2}$$

where A, B, Q, and R are constant matrices of compatible dimensions and appropriate definiteness, and σ is an arbitrary positive constant. The problem of interest is to find an optimal control law such that (2) is minimized subject to (1). Because the integrand of (2) is unbounded when the terminal time approaches infinity, the method of derivation of the conventional optimal linear regulator cannot be directly employed without some prior mathematical modifications. With the use of the concept of cost equivalence, it is possible to state and prove the following result.

Theorem: The optimal control law which minimizes (2) subject to (1) is given by

$$u^* = -R^{-1}B'Sx \tag{3}$$

where the symmetric matrix S is the unique positive definite solution of the algebraic matrix Riccati equation

$$A'S + SA + 2\sigma S - SBR^{-1}B'S + Q = 0 \tag{4}$$

Moreover, the real parts of all the eigenvalues of the resulting closed-loop system $(A - BR^{-1}B'S)$ are less than $-\sigma$.

Proof: It is well known that if (A,B) is a controllable pair, $(A + \sigma I,B)$ is also controllable for any σ. From a result by Wonham,[5] there always exists a feedback control law of the form

$$u = -Kx \tag{5}$$

such that $(A + \sigma I - BK)$ is a stability matrix, that is, the real parts of all the eigenvalues of $(A - BK)$ are less than $-\sigma$. Applying (5) to (1) and (2) gives $[A - BK,e^{2\sigma t}(Q + K'RK)]$, which, in view of the previous lemma, is equivalent to $[A + \sigma I - BK,Q + K'RK]$ in which

$$\dot{z} = (A + \sigma I - BK)z, \; z(0) = x_0 \tag{6}$$

$$J = \int_0^\infty z'(Q + K'RK)zdt \tag{7}$$

Using now the conventional theory of optimal linear regulator, the control law that minimizes (7) subject to (6) is given by

$$\hat{u} = -R^{-1}B'Sz \tag{8}$$

where the symmetric S is given by (4). Furthermore, the matrix $(A + \sigma I - BR^{-1}B'S)$ is asymptotically stable, and, consequently, the real parts of all the eigenvalues of the closed-loop system are less than $-\sigma$.

It should be noted that the optimal control law of (8) is a linear feedback of the vector z instead of the state vector x. To complete the proof of the theorem, it is required to show that both (3) and (8) result in identical costs. Applying (3) to (1) and (2) leads to $[A - BR^{-1}B'S,e^{2\sigma t}(Q + SBR^{-1}B'S)]$ which, with the use of the previous lemma, is equivalent to $[A + \sigma I - BR^{-1}B'S,Q + SBR^{-1}B'S]$. Indeed, the latter is the optimum representation of (6) and (7), and hence (3) is the optimal control law minimizing (2) subject to (1).

It is of interest to note that an efficient algorithm for the numerical solution of (4) can be found in Ref. 6.

Linear Time-Varying Systems

The theory outlined in the previous sections is equally applicable to cases when the system (1) is time-varying and the terminal time in (2) is finite; that is, when (1) and (2) have the form

$$\dot{x} = A(t)x + B(t)u \tag{9}$$

$$J = \int_{t_0}^{t_f} e^{2\sigma t}(x'Qx + u'Ru)dt \tag{10}$$

Utilizing the arguments given previously with a slight modification, it is readily shown that the optimal control minimizing (10) subject to (9) is given by

$$u^* = -R^{-1}B'S(t)x \tag{11}$$

where the symmetric matrix $S(t)$ satisfies the matrix Riccati equation

$$\dot{S}(t) = -[A(t) + \sigma I]'S(t) - S(t)[A(t) + \sigma I] + S(t)B(t)R^{-1}B'(t)S(t) - Q,\ S(t_f) = 0 \tag{12}$$

On the other hand, using the theory of optimal linear regulator, the optimal control is

$$\hat{u} = -e^{-2\sigma t}R^{-1}B'(t)\hat{S}(t)z \tag{13}$$

where the symmetric matrix $\hat{S}(t)$ satisfies the following matrix Riccati equation:

$$\dot{\hat{S}}(t) = -A'(t)\hat{S}(t) - \hat{S}(t)A(t) + e^{-2\sigma t}\hat{S}(t)B(t)R^{-1}B'(t)\hat{S}(t) - e^{2\sigma t}Q$$

$$\hat{S}(t_f) = 0 \tag{14}$$

It can be demonstrated that (13) and (14) are identical to (11) and (12) by employing the nonsingular transformation $\hat{S}(t) = e^{2\sigma t}S(t)$. This identification also demonstrates the time invariance of the optimal control law as given by (3) and (4) for the time-invariant system when the terminal time approaches infinity.

An Example

Consider a third-order unstable system given by

$$A = \begin{bmatrix} 1 & 2 & -3.0 \\ -4 & 5 & -0.6 \\ 7 & 8 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0.1 & 0 & 2 \\ 0.0 & 10 & 0 \\ 0.0 & 0 & 100 \end{bmatrix}$$

Taking the matrix $R = I$ and the matrix Q arbitrarily to be 0.1I with $\sigma = 0$, the eigenvalues of the resulting closed-loop system are -30.63 and $-4.58 \pm j2.33$. Using the same Q and R as before but with $\sigma = 3$, the eigenvalues of the resulting closed-loop system become -33.70 and $-9.90 \pm j2.30$. It can be seen that for a given set of weighting matrices Q and R, the design based on the exponentially time-weighted performance index does give a fast and well-damped system response. However, as pointed out in Ref. 1, although the use of exponentially time-weighted performance indices provides a satisfactory means of shifting the dominant eigenvalue of the system, the remaining modes of the system may be quite oscillatory. It appears that the applicability of the exponentially time-weighted performance indices is greatly enhanced when they are used in conjunction with a design technique such as the root-square locus.[7]

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Upwash Interference on an Oscillating Wing in Slotted-Wall Wind Tunnels

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Introduction

THE wall interference on a stationary wing in subsonic flow in a wind tunnel with ventilated walls has been studied extensively (see a summary in Ref. 1). However, the calculation of the interference on an oscillating airfoil has been limited because of the complexity of the problem. Experimental evidence indicates[2] that the interference effects on an oscillating airfoil may be large in slotted wall tunnels; however, calculations of this interference are limited to certain special cases.[2]

This paper presents the upwash interference on an oscillating wing in a slotted-wall tunnel for all frequencies. The formulation is based on the small-wing theory with a relationship between the steady acceleration and unsteady velocity potentials. An analytical solution is given for the upwash interference in a circular and in a rectangular tunnel with solid side walls.

Analysis

If the flow is oscillating with the angular frequency ω due to the harmonic motion of a wing, the perturbation potential may be written as $\phi = \phi_k(x,y,z)e^{i\omega t}$. The linearized equation for ϕ of a thin wing becomes

$$\beta^2 \frac{\partial^2\phi_k}{\partial x^2} + \frac{\partial^2\phi_k}{\partial y^2} + \frac{\partial^2\phi_k}{\partial z^2} - 2ikM^2\frac{\partial\phi_k}{\partial x} + k^2M^2\phi_k = 0 \tag{1}$$

where $k = \omega L/U$ is the reduced frequency and $\beta^2 = 1 - M^2$. U and M are freestream velocity and Mach number, respectively. All lengths are nondimensionalized by a tunnel characteristic length L.

From the definition of the acceleration potential

$$\psi_k = (\partial\phi_k/\partial x + ik\phi_k) \tag{2}$$

the inverse relation for ϕ_k can be obtained by integration with respect to x since ϕ_k vanishes far upstream at $x = -\infty$

$$\phi_k = \int_0^\infty e^{-ik\xi}\,\psi_k(x - \xi,y,z)d\xi \tag{3}$$

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